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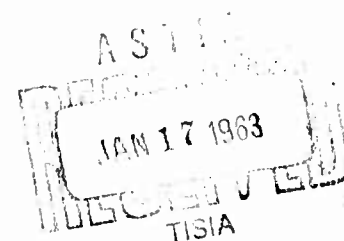
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Studies in Generalized Random Walks

I. Distribution Functions and Moments.



A. William Kratzke

Mathematics Research

November 1962

STUDIES IN GENERALIZED RANDOM WALKS

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Abstract

In the simple one-dimensional random walk a particle moves along a line in unit steps, either forward or backward. Consider the motion of the particle if the step lengths are not constant, but rather variables that depend on the position of the particle at each step. Such a generalization of the random walk is the subject of these studies.

The motion of the particle is studied in some detail. If x is the position of the particle before a move, the particle has two alternatives; it may move to $C_1 + C_2x$ or C_3x where C_1 , C_2 , and C_3 are constants subject to certain constraints. Several theorems are proved which specify just what points are accessible to the particle in a finite number of steps as well as for the limiting case of infinite steps.

A fixed probability law which makes the walk random is assumed, i.e. the two alternatives described above are assigned probabilities p and q and $p + q = 1$. Then the distribution function for the particle position after a given number of steps is determined. Finally there is derived a body of results on the moments. These latter results are found without knowledge of the distribution functions.

The random walk has served as a model or analogue for a wide variety of problems in fields as unrelated as colloid chemistry and gambling. The problem of random flights was considered by K. Pearson around the turn of the century. In that problem the distribution of position of a particle after it had suffered a sequence of n displacements was sought. This is the same as seeking the distribution of the sum of n random vectors, each vector being characterized by a probability distribution. Earlier Lord Rayleigh had studied the problem of finding the distribution of the composition of n periodic vibrations of unit amplitude and random phases. Later Smoluchowski and Einstein found the distribution of particles in Brownian motion. In 1923, Wiener discussed this stochastic process rigorously. The salient feature of each of these problems is the distribution of a sum of random variables, each random variable being characterized by an assigned distribution.

Since sums of random variables play a dominant role in such analyses, it is natural to expand on the applications of the methods and include all problems that involve additive quantities such as energy, momentum, length, financial gain, population, etc.

All such problems may be thought of as elaborations (albeit complicated) on the simplest one-dimensional random walk, which will be described now.

A particle moves along a straight line in steps at times $t = 1, 2, 3, \dots$, each step being of unit length and taken either forward or backward with

equal probability, $1/2$. After n steps the particle could be at any of the points, $-n, -n+1, \dots, -1, 0, +1, \dots, n-1, n$. The probability that the particle is at position m after n steps is the familiar Bernoulli distribution. This formulation is the model for one of the simplest of gambling problems, viz. that of the gambler who wins or loses one unit of money at each play of the game and he wins and loses with the same probability, $1/2$.

The first simple generalization of the one-dimensional random walk results when the probability of a forward step (winning) is not the same as the probability of a backward step (losing), but rather p and q respectively. Further generalizations result when problems in higher dimensions are considered and problems still more complicated result when reflecting and absorbing barriers are introduced. Again, appealing to the gambler analogue, an absorbing barrier at the origin represents the gambler's ruin point and problems involving probability of ruin and duration of the game (number of steps or time till ruin or absorption at a barrier) become interesting and important. These problems have now become 'classical' and are well documented.

One property tacitly implied in all of the work discussed above is independence. The 'step lengths' have been taken to be statistically independent and in most cases identically distributed. The generalization of the random walk which is discussed here imposes a special kind of dependence among the random variables of the process. Specifically, the length of a forward step is made to be different from that of a backward step and these step lengths depend on the position of

the particle before the step is taken. The probabilities of moving forward or backward are constants, p and $q (= 1 - p)$.

These studies are presented in three parts. The first part, under this cover, treats the problem of finding the distribution function of the particle position after n steps and the limiting case for large n . Moments for particle position are derived exactly, both for finite numbers of steps and for the limiting case.

Part two is a study of the absorption time or the number of steps needed for a particle to exceed a pre-assigned level. That number or time is a random variable and its expected value is studied in that paper.

In part three a number of applications are presented and discussed. Some of these applications provided the motivation for these studies and the formulation of these phenomena as random walks is shown in detail.

The author acknowledges with thanks the instructive discussions he has had with L. Takács, S. Karlin, J. Gani, R. Pyke, A. Marshall, and J. Tysver on these problems.

I. Distribution Functions and Moments.

1. Introduction.

Consider the following random walk:

A particle undergoes excursions along the x-axis in such a way that the position after n steps depends on its position after $n-1$ steps. The particle at each position is subject to one of two impulses and its motion is governed by a fixed probability law. Explicitly,

$$x_n = \begin{cases} C_1 + C_2 x_{n-1} & \text{with probability } p \\ C_3 x_{n-1} & \text{with probability } q (= 1 - p), \end{cases} \quad (1)$$

where x_j is the particle position after j steps and C_1 , C_2 , and C_3 are constants subject to a set of constraints.

Before proceeding to the analyses of this random walk that lead to equations for distribution functions and the moments, it seems relevant to discuss the nature of the motion itself.

2. Motion of a particle in the random walk.

The admissible values that the constants, C_1 , C_2 , and C_3 may assume must first be specified. Later on certain more stringent constraints will be treated but for now the following inequalities will suffice:

$$\begin{aligned} C_1 &> 0 \\ 0 &< C_2 < 1 \\ 0 &< C_3 < 1. \end{aligned} \quad (2)$$

Now consider the possible positions a particle may occupy after n steps, having started at x , when

$$0 \leq x \leq \frac{C_1}{1 - C_2}.$$

Henceforth, the closed interval, $[0, \frac{C_1}{1-C_2}]$ will be denoted by I .

Since there are two alternatives at each step, there are 2^n possible positions the particle may occupy after n steps. For example, after one step, two steps, etc. the particle may be at

$$\begin{aligned}
 & C_1 + C_2 x \quad \text{or} \\
 & C_3 x \qquad \qquad \qquad \text{(after one step),} \\
 & C_1(1 + C_2) + C_2^2 x, \\
 & C_1 + C_2 C_3 x, \qquad \qquad \qquad \text{(after two steps),} \\
 & C_1 C_3 + C_2 C_3 x, \quad \text{or} \\
 & C_3^2 x \\
 & \vdots \\
 & \text{etc.}
 \end{aligned} \tag{3}$$

These points are determined merely by repeated application of equation (1).

S. Karlin, in private correspondence, has pointed out that all such positions may be represented by a single expression. Consider the sequence $\alpha = \{a_1, a_2, a_3, \dots, a_n\}$ where $a_k = 0$ or 1 . There are 2^n such sequences of length n and they generate the particle positions as follows. If $t(\alpha)$ is the particle position corresponding to the sequence, α ,

$$t(\alpha) = C_1 \sum_{j=1}^n a_j C_2^{\sum_{i=1}^{j-1} a_i} C_3^{j-1 - \sum_{i=1}^{j-1} a_i} + C_2^{\sum_{i=1}^n a_i} C_3^{n - \sum_{i=1}^n a_i} x. \tag{4}$$

Note that in the limit, i.e. for infinite sequences, the dependence on the initial position vanishes. Furthermore, for $x = 0$, all sequences

may be considered infinite with all terms zero after the n th term for the n - length finite sequences; $\{a_1, a_2, \dots, a_n\}$ becomes $\{a_1, a_2, \dots, a_n, 0, 0, \dots\}$. Also observe that if a particle's initial position is in the interval, I , it can never leave that interval. In particular, an infinite number of forward steps ($x \rightarrow C_1 + C_2 x$) and an infinite number of backward steps ($x \rightarrow C_3 x$) yield terminal positions

$$\lim_{n \rightarrow \infty} (C_1 \sum_{i=0}^n C_2^i + C_2^{n+1} x) = \frac{C_1}{1 - C_2} \quad \text{and} \quad (5)$$

$$\lim_{n \rightarrow \infty} C_3^n x = 0$$

respectively.

Thus far the discussion of the motion has been concerned with the points that may be reached from an initial position in a given number of steps. The points attainable in n steps from an initial position, x , are given by (4) and the 2^n n - length sequences of zeros and ones, α . Just as pertinent to these analyses is the question of whether or not a given point in the interval, I , can be reached from any other point in the interval, with no restriction on the number of steps that it may take. This is the same as asking if one can start at x and arrive at x' , where x and x' are arbitrary. If x' cannot be reached from x in a finite number of steps, how close one can come to x in a finite number of steps is also interesting.

To consider the above problems it is necessary to categorize the constraints. Those constraints given by (2) still hold but the relationship between C_2 and C_3 must be considered. Two distinct possibilities

yield slightly different results. Those two cases are given by

$$\begin{aligned} C_2 + C_3 &\geq 1 \quad \text{and} \\ C_2 + C_3 &< 1. \end{aligned} \tag{6}$$

It is also expedient to recast the description of the particle motion in the language of operators. Let T_1 , T_2 , T_1^{-1} , and T_2^{-1} be operators on points of the interval, I , defined by

$$\begin{aligned} T_1 x &= C_1 + C_2 x \\ T_2 x &= C_3 x \\ T_1^{-1} x &= \frac{x - C_1}{C_2} \\ T_2^{-1} x &= \frac{x}{C_3}. \end{aligned} \tag{7}$$

Of course, the inverse operators, T_1^{-1} and T_2^{-1} , seem superfluous to the description of the motion since the random walk model described here does not admit of steps that carry x to $\frac{x - C_1}{C_2}$ or x to $\frac{x}{C_3}$. Moreover, not all of the points of the interval, I , are in the domains of T_1^{-1} and T_2^{-1} if it is required that the ranges of these operators both be subsets of I . The usefulness of T_1^{-1} and T_2^{-1} will be made clear in the sequel and their inclusion among definitions here is merely for completeness.

Let the operator T be a generic operator indicating either T_1 or T_2 . Similarly let T^{-1} mean either T_1^{-1} or T_2^{-1} . Finally, T^m means some sequence of operations using T_1 's and T_2 's and T^{-m} means some sequence of operations using T_1^{-1} 's and T_2^{-1} 's.

Now note the following property of sequences of operations on points of I . For all x and x' in I ,

$$|T^m x - T^m x'| \leq c^m |x - x'| \leq c^m \frac{C_1}{1 - C_2} \quad (8)$$

where $c = \max(C_2, C_3)$. The constraints given by (2) guarantee that the difference may be made small. The inequality (8) follows immediately by direct calculation.

$$|T^m x - T^m x'| = |C_2^k C_3^{m-k} (x - x')| = C_2^k C_3^{m-k} |x - x'| \leq C_2^k C_3^{m-k} \frac{C_1}{1 - C_2} \quad (9)$$

for some $k (0 \leq k \leq m)$ depending on the sequence of operations. In fact, k is just the number of times T_1 appears in the sequence T^m and, of course, $m - k$ is the number of times T_2 appears in T^m .

Returning to the discussion of the approach of a point from another point, the following theorems show that if x is an initial point, one can come arbitrarily close to x' , a given point in a subset of I (the subset is determined by the constraints (6) and x' can be reached in the limit.).

Theorem 1. $C_1 > 0$, $0 < C_2, C_3 < 1$, $C_2 + C_3 \geq 1$.

$$T_1 x = C_1 + C_2 x.$$

$$T_2 x = C_3 x.$$

$$Tx = T_1 x \text{ or } T_2 x.$$

$$x \in I, x' \in I, I = [0, \frac{C_1}{1 - C_2}].$$

Given an $\epsilon > 0$, there exists an $M(\epsilon)$ and sequences of operations, T , such that if $m > M$,

$$|T^m x - x'| < \epsilon$$

Proof: Choose M so that

$$c^M < \frac{1 - c_2}{c_1} \epsilon \quad (c = \text{maximum } (c_2, c_3))$$

$$\text{if } \epsilon < \frac{c_1}{1 - c_2} \quad \text{or } M = 0 \quad \text{if } \epsilon \geq \frac{c_1}{1 - c_2} .$$

Either $T_1^{-1}x' \in I$ or $T_2^{-1}x' \in I$ or both. Choose an inverse operator (T_1^{-1} or T_2^{-1}) so that $T^{-1}x' \in I$. Next choose a second inverse operator so that $T^{-1}T^{-1}x' \in I$, and so on, until finally,

$$T^{-m}x' \in I.$$

From (9) if $m > M$,

$$\begin{aligned} |T^m x - x'| &= |T^m x - T^m T^{-m} x'| \leq c^m |x - T^{-m} x'| \\ &\leq \frac{c^m c_1}{1 - c_2} < \frac{c^M c_1}{1 - c_2} < \epsilon. \end{aligned} \quad (10)$$

Therefore, if x' cannot be reached in a finite number of steps, it may always be reached as a limit point of a $T^m x$ sequence.

The sequence of transformations used above need some closer scrutiny, since the results of Theorem 1 may otherwise be misleading. First note that the order of operations in the T^m sequence in (10) is the reverse of that in the T^{-m} sequence. Since neither T_1 and T_2 nor T_1^{-1} and T_2^{-1} commute, the order of operations must be chosen this way. Then, if E is the identity transformation, i.e. $Ex = x$,

$$\dots T_k T_j T_i T_i^{-1} T_j^{-1} T_k^{-1} \dots x = Ex = x. \quad (11)$$

Now (10) implies some sort of convergence of sequences of operators, T^m, T^{m+1} , etc. and convergence depends on the definition of the sequences. The distinction is obvious if the following illustrations are considered.

Let superscripts indicate operator labels and let $(T)^m$ indicate m applications of T 's. Then consider the following sequence.

$$\begin{aligned} T^1 x &= (T)x \\ T^2 T^1 x &= (T)^2 x \\ T^3 T^2 T^1 x &= (T)^3 x \\ &\vdots \\ T^m T^{m-1} \dots T^1 x &= (T)^m x. \end{aligned} \tag{12}$$

The limit of $(T)^m x$ always yields a point in the interval but one more application of either T_1 or T_2 removes a particle from that point. This means that one never reaches a point x_0 such that $T_1 x_0 = x_0$ and $T_2 x_0 = x_0$. So the sequences defined by (12) do not converge.

Now consider the kind of sequences used in Theorem 1.

$$\begin{aligned} T^1 x &= (T)x \\ T^1 T^2 x &= (T)^2 x \\ &\vdots \\ T^1 T^2 T^3 \dots T^{m-1} T^m x &= (T)^m x \end{aligned} \tag{13}$$

There it was shown that for every point, x' , in the interval, I , there exists a sequence of operations on an arbitrary starting point, whose limit is x' . In fact, the method for constructing that sequence yields the arrangement (13). That every sequence chosen in the manner that T^m appears in Theorem 1 converges is demonstrated in

Theorem 2. Every sequence (13) converges uniformly in x .

Proof: It suffices to show that (13) is a Cauchy sequence.

If $m > n$, (9) gives

$$\begin{aligned} |T^m x - T^n x| &= |T^n T^{m-n} x - T^n x| \leq c^n |T^{m-n} x - x| \\ &\leq c^n \frac{c_1}{1 - c_2} \end{aligned} \quad (14)$$

where c again is $\max(C_2, C_3)$

and

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} |T^m x - T^n x| = 0 \quad [\text{uniformly}]. \quad (15)$$

The other case considered is characterized by the constraint, $C_2 + C_3 < 1$. Here the motion is not so simple and not all the points in the interval, I , are accessible points. Note that applying T_1 to the interval, I (the transformations may now be thought of as mapping sets into sets) and T_2 to the interval, yields

$$\begin{aligned} T_1[0, \frac{c_1}{1 - c_2}] &= [c_1, \frac{c_1}{1 - c_2}] \\ T_2[0, \frac{c_1}{1 - c_2}] &= [0, \frac{c_1 c_3}{1 - c_2}] \end{aligned} \quad (16)$$

The union of these sets does not cover the interval. The open interval, $(\frac{c_1 c_3}{1 - c_2}, c_1)$ is not covered. The union of two applications of T_1 and T_2 on I leaves uncovered the intervals, $(\frac{c_1 c_3}{1 - c_2}, c_1)$, $(\frac{c_1 c_3^2}{1 - c_2}, c_1 c_3)$, and $(c_1(1 + \frac{c_2 c_3}{1 - c_2}), c_1(1 + c_2))$. Further applications of T_1 and T_2 leave additional intervals uncovered in the same way that a Cantor set is constructed. In fact, if $C_1 = \frac{2}{3}$, $C_2 = \frac{1}{3}$, and $C_3 = \frac{1}{3}$, this succession of steps provides a method for constructing the

Cantor ternary set. Another way of showing this is by noting the constraints on the inverse operators used in Theorem 1. The only points, x' , for which either $T_1^{-1}x'$ or $T_2^{-1}x'$ is admissible as a first backward step are in the set

$$[0, \frac{c_1 c_3}{1 - c_2}] \cup [c_1, \frac{c_1}{1 - c_2}].$$

Having removed the interval $(\frac{c_1 c_3}{1 - c_2}, c_1)$, the only points eligible to be transformed backward another step are in

$$[0, \frac{c_1 c_3^2}{1 - c_2}] \cup [c_1 c_3, \frac{c_1 c_3}{1 - c_2}] \cup [c_1, c_1(1 + \frac{c_2 c_3}{1 - c_2})] \cup [c_1(1 + c_2), \frac{c_1}{1 + c_2}].$$

Continuing in this way, it is clear that the only points from which one may start in a sequence of backward steps belong to a Cantor set, \mathcal{K} . The companion theorem to Theorem 1, is then,

Theorem 3. $c_1 > 0$, $0 < c_2, c_3 < 1$, $c_2 + c_3 < 1$

$x' \in \mathcal{K}$, the Cantor set constructed by the method described above.

For all x in I , one can come arbitrarily close to x' in a finite number of steps and can always reach x' in an infinite number of steps.

Proof: With the suitable restrictions described in the text, the proof follows immediately from Theorem 1.

Thus, when $c_2 + c_3 < 1$ and x' is in the open interval, $I - \mathcal{K}$, there is no sequence of operations, finite or infinite, that will terminate at x' except the trivial case, i.e., $x = x'$ and the sequence of operations is of zero length.

Observe that x' can be reached from x for all x' in I so the convergence to x' is uniform in x (with the tacit assumption that x' is in the proper interval).

Finally, the next two theorems show how points in the interval, I , may be represented. The connection between the behavior of a particle suffering impulses given by the transformations, T_1 and T_2 , and the representation of terminal points is the main result. Theorems 1, 2, and 3 and the representation (4) are closely related and this relation is made obvious by these two theorems. Without loss of generality, the initial point, x , is taken to be zero.

Theorem 4. $C_1 > 0$, $0 < C_2, C_3 < 1$, $C_2 + C_3 \geq 1$.

$$x' \in I = [0, \frac{C_1}{1 - C_2}].$$

There exists a sequence $\alpha = \{a_1, a_2, \dots\}$, where $a_k = 0$ or 1 , such that

$$x' = C_1 \sum_{n=1}^{\infty} a_n C_2^{n-1} C_3^{n-1-\sum_{i=1}^{n-1} a_i} \quad (17)$$

If x' is one of the points that can be reached from zero in a finite number of steps, the result is obvious. Suppose x' cannot be reached from zero in a finite number of steps. Theorem 1 asserts that there exists a sequence of transformations such that

$$\lim_{m \rightarrow \infty} T^m 0 = x' \quad (18)$$

It is clear that the transformations, T_1 and T_2 , correspond to one and zero respectively in the sequence α . For example,

$$\begin{aligned}
 T_1 O &= C_1 & \alpha &= \{1, 0, 0, \dots\} \\
 T_2 O &= 0 & \alpha &= \{0, 0, 0, \dots\} \\
 T_1 T_1 O &= C_1(1 + C_2) & \alpha &= \{1, 1, 0, \dots\} \\
 T_2 T_1 O &= C_3 C_1 & \alpha &= \{0, 1, 0, \dots\} \\
 T_1 T_1 T_1 O &= C_1(1 + C_2 + C_2^2) & \alpha &= \{1, 1, 1, 0, 0, \dots\} \\
 T_2 T_1 T_1 O &= C_1 C_3(1 + C_2) & \alpha &= \{0, 1, 1, 0, 0, \dots\} \\
 & \text{etc.}
 \end{aligned} \tag{19}$$

Therefore, any point in I can be represented by (17) if the sequence, α , is made to correspond to the infinite sequence of transformations that yield (18).

One special case merits consideration, viz. when $C_2 + C_3 = 1$. When this situation obtains, the sequence of transformations whose limit is x' is unique (except for the fact that $T_1 T_2 T_2 T_2 \dots$ is equivalent to $T_2 T_1 T_1 T_1 \dots$ just as $0.011\dots$ is equivalent to $0.100\dots$ in the binary expansion of a number in the unit interval). When $C_2 + C_3 = 1$ each backward step in Theorem 1 is determined uniquely since $T_1^{-1}x'$ and $T_2^{-1}x'$ can never be in I simultaneously. For example, if $C_1 < x' < \frac{C_1}{1 - C_2}$ the first inverse transformation must be T_1^{-1} . Each step is thus determined uniquely. This also specifies the sequence, α , uniquely. If the first inverse transformation is T_1^{-1} , $a_1 = 1$; if the transformation is T_2^{-1} , $a_1 = 0$. Then a_2, a_3 , etc. are chosen the same way and (17) gives a representation for x' that is unique. An obvious special case arises when $C_1 = C_2 = C_3 = 1/2$.

Then (17) gives the binary expansion for numbers in the unit interval.

A number written in binary form is just the sequence, α ;

i.e., $x' = 0.a_1a_2a_3\dots$. If $C_2 + C_1 = 1$ and $C_2 + C_3 > 1$, (17)

gives another way of representing a number in the unit interval using a sequence of only zeros and ones, although the representation may not be unique.

Theorem 5. $C_1 > 0$, $0 < C_2, C_3 < 1$, $C_2 + C_3 < 1$.

$x' \in I - \mathcal{K}$.

There exists a sequence $\alpha = \{a_1, a_2, \dots\}$, where $a_k = 0$ or 1 , such that x' is represented by (17).

Proof: The proof is the same as that for Theorem 4.

These convergence theorems and representation theorems provide some insight into the nature of the motion of a particle subject to the impulses described in (1).

3. Distribution Functions.

Recall that the motion of a particle in the interval, $[0, \frac{c_1}{1-c_2}]$, is governed by the fixed probability law described by (1). The distribution function characterizing the particle position after each step is, then, determined by that law. Let $F_n(t)$ be the conditional probability distribution of the particle position after n steps.

$$\left. \begin{aligned} F_n(t) &= \Pr\{x_n \leq t | x_0 = x_0\} \\ F_n(t) &\equiv 1, \quad t \geq \frac{c_1}{1-c_2} \\ F_n(t) &= 0, \quad t \leq 0. \end{aligned} \right\} \quad n = 0, 1, 2, \dots \quad (20)$$

The conditioning on x_0 has been here suppressed since it will always be apparent when it applies.

To derive expressions for F_n , note that if x is an arbitrary point in $I (= [0, \frac{c_1}{1-c_2}])$, a particle can reach x at the n th step in one of two ways. It can have been at $\frac{x-c_1}{c_2}$ after $n-1$ steps and advanced to x with probability, p , or it can have been at $\frac{x}{c_3}$ after $n-1$ steps and moved to x with probability, q . Therefore, the distribution functions, F_n and F_{n-1} are related by

$$F_n(x) = pF_{n-1}\left(\frac{x-c_1}{c_2}\right) + qF_{n-1}\left(\frac{x}{c_3}\right). \quad (21)$$

The arguments of the probability measures on the right side of this expression are the sets, $[0, \frac{x-c_1}{c_2})$ and $[0, \frac{x}{c_3})$, and they are ordered by the inclusion relation

$$[0, \frac{x-c_1}{c_2}) \subset [0, \frac{x}{c_3}).$$

This may seem to make (21) intuitively unappealing since the probability that the particle is at a point less than $\frac{x}{c_3}$ includes the probability that it is at a point less than $\frac{x-c_1}{c_2}$. But the arguments

used above also show that the differential probability that a particle is at x after n steps is

$$dF_n(x) = p dF_{n-1}\left(\frac{x - C_1}{C_2}\right) + q dF_{n-1}\left(\frac{x}{C_3}\right). \quad (22)$$

Then (21) follows immediately by

$$\int_0^x dF_n(t) = p \int_0^x dF_{n-1}\left(\frac{t - C_1}{C_2}\right) + q \int_0^x dF_{n-1}\left(\frac{t}{C_3}\right) \quad (23)$$

For small n , it is easy to determine $F_n(x)$. Repeated application of (21) yields

$$\begin{aligned} F_n(x) &= p F_{n-1}\left(\frac{x - C_1}{C_2}\right) + q F_{n-1}\left(\frac{x}{C_3}\right), \\ &= p^2 F_{n-2}\left(\frac{x - C_1(1 + C_2)}{C_2^2}\right) + pq F_{n-2}\left(\frac{x - C_1}{C_2 C_3}\right) \\ &\quad + pq F_{n-2}\left(\frac{x - C_1 C_3}{C_2 C_3}\right) + q^2 F_{n-2}\left(\frac{x}{C_3^2}\right) \\ &\quad \vdots \\ &= p^n F_0\left(\frac{x - C_1 \frac{1 - C_2^n}{1 - C_2}}{C_2^n}\right) + \dots + q^n F_0\left(\frac{x}{C_3^n}\right). \end{aligned} \quad (24)$$

Every term on the right side of (24) involves $F_0(\cdot)$ and since

$$\begin{aligned} F_0(x) &= 0 & x < x_0 \\ F_0(x) &= 1 & x \geq x_0, \end{aligned}$$

all one needs to do to determine $F_n(x)$ is check the arguments of the individual terms and substitute zero or one according as the argument is less than or greater than x_0 , the conditioning initial point. Some simplification obtains if the terms are examined after each application

of (21) since, by (20)

$$F_k(x) = 1 \quad x \geq \frac{C_1}{1 - C_2} \quad 0 \leq k \leq n - 1.$$

$$= 0 \quad x \leq 0.$$

Now consider the limiting distribution, that is, the distribution after an infinite number of steps. As in the discussion of the motion of the particle (Section 2), the constraints imposed by the values of C_1 , C_2 , and C_3 become important. Furthermore, the relationships among p , q , C_2 , and C_3 enter the discussion.

If $C_2 + C_3 < 1$, the limiting distribution may be found by examining the representation of a point given by Theorems 3 and 5 and noting that the transformation T_1 corresponds to a move that has probability, p , of happening and T_2 corresponds to a move that has probability, q . Theorem 5 shows that if $x \in K$, the Cantor set constructed in Section 2, x may be represented by equation (17). For $C_2 + C_3 \leq 1$, x is monotonic in α .

$$\alpha_1 > \alpha_2 \implies x_1 > x_2.$$

By $\alpha_1 > \alpha_2$ is meant that if the α 's are thought of as binary numbers, the relation between those numbers determines the ordering of the α 's. For example,

$$\{1,0,0,\dots\} > \{0,1,0,0,\dots\} > \{0,0,1,0,0,\dots\}.$$

Notice that this monotonicity obtains only when $C_2 + C_3 \leq 1$, for only then are the representations unique (see Section 2).

Now, when $x \in \mathcal{K}$, it follows that

$$x = c_1 \sum_{n=1}^{\infty} a_n c_2^{i=1} c_3^{n-1-\sum_{i=1}^{n-1} a_i} \quad \text{and} \quad (25)$$

$$F(x) = q \sum_{n=1}^{\infty} a_n p^{i=1} q^{n-1-\sum_{i=1}^{n-1} a_i}$$

If $x \notin \mathcal{K}$, it is necessary to determine which interval contains x . x must necessarily be a member of one of the intervals removed in constructing \mathcal{K} . If $x \in (a, b)$, then, because of right continuity and monotonicity of F , surely

$$F(x) = F(a) \quad (26)$$

and $a \in \mathcal{K}$, so $F(a)$ is found by (25). Finding the interval to which x belongs involves the inverse operator procedure outlined in Section 2. If neither $T_1^{-1}x$ nor $T_2^{-1}x$ exists,

$$F(x) = F\left(\frac{c_1 c_3}{1 - c_2}\right).$$

If either $T_1^{-1}x$ or $T_2^{-1}x$ exists, one tries the next inverse operation. Say $T_1^{-1}x$ exists, but neither $T_1^{-1}T_1^{-1}x$ nor $T_2^{-1}T_1^{-1}x$ exists. Then by solving

$$\frac{x' - c_1}{c_2 c_3} = \frac{c_1 c_3}{1 - c_2}$$

for x' ,

$$F(x) = F(x') = F\left(c_1 \left(1 + \frac{c_2 c_3^2}{1 - c_2}\right)\right).$$

Continuing in this way, it is always possible to find the interval to which x belongs and since F is monotone it is surely constant in that interval and equal to $F(a)$ where a is the left end point of the interval.

Next, consider the limiting distribution when $C_2 + C_3 = 1$. In this case, every x in $[0, \frac{C_1}{1-C_2}]$ may be expressed by (17) and this representation is unique. For all x , $F(x)$ is given by (25). However, some special cases are interesting and admit of expressions simpler than (25).

If $p = C_2$ and $q = C_3$, observe from (25) that

$$F(x) = \frac{q}{C_1} x = \frac{C_3}{C_1} x. \quad (27)$$

If $p < C_2, q > C_3$, and $C_2 + C_3 = 1$, $F(x)$ becomes concave and, similarly, if $p > C_2, q < C_3$, and $C_2 + C_3 = 1$, $F(x)$ becomes convex.

Figure 1 illustrates three cases:

Curve A: $p = C_2, q = C_3$

Curve B: $p < C_2, q > C_3$

Curve C: $p > C_2, q < C_3$

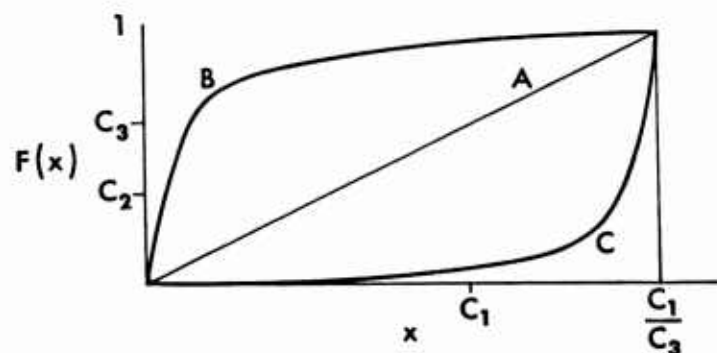


Fig. 1

For purposes of application, $F(x)$ may be approximated by noting the agreement between (25) and:

$$\begin{aligned} F(x) &= \left(\frac{C_3}{C_1} x\right)^\beta \quad 0 \leq x \leq C_1, \quad \beta = \frac{\ln q}{\ln C_3} \\ &= 1 - \left(1 - \frac{C_3}{C_1} x\right)^\delta \quad C_1 \leq x \leq \frac{C_1}{C_3}, \quad \delta = \frac{\ln p}{\ln C_2}. \end{aligned} \quad (28)$$

For $x = C_1, C_1 C_3, C_1 C_3^2, \dots, C_1 C_3^k, \dots$, $F(x) = \left(\frac{C_3}{C_1} x\right)^\beta$ agrees exactly with (25)

and for $x = C_1, \frac{C_1}{C_3}(1 - C_2^2), \frac{C_1}{C_3}(1 - C_2^3), \dots, \frac{C_1}{C_3}(1 - C_2^k), \dots$,

$F(x) = 1 - \left(1 - \frac{C_3}{C_1} x\right)^\delta$ agrees with (25).

Finally, a convenient check is available for all of the expressions for F . The equation for the limiting distribution which corresponds to (21) is

$$F(x) = pF\left(\frac{x - C_1}{C_2}\right) + qF\left(\frac{x}{C_3}\right), \quad (29)$$

again with the obvious conditions,

$$\begin{aligned} F(x) &= 0 & x &\leq 0 \\ &= 1 & x &\geq \frac{C_1}{1 - C_2}. \end{aligned}$$

Only those distributions for which $C_2 + C_3 = 1$ are considered. When $C_2 + C_3 < 1$, the same arguments apply and that case lends nothing to the discussion.

Suppose $C_2 + C_3 = 1$. Then if $x < C_1$, the first element of the sequence, α , corresponding to x is a zero. If $x > C_1$, the first element is a one. When $x = C_1$,

$$C_1 = \{1, 0, 0, \dots\} = \{0, 1, 1, \dots\}.$$

Now consider (25). If $x < C_1$,

$$x = \{0, a_2, a_3, \dots\}$$

$$\frac{x}{C_3} = \{a_2, a_3, \dots\}$$

$$F(x) = qF\left(\frac{x}{C_3}\right).$$

$$\begin{aligned} F(x) &= q(a_2 q + a_3 p^{a_2} q^{2-a_2} + a_4 p^{a_2+a_3} q^{3-(a_2+a_3)} + \dots) \\ &= q^2(a_2 + a_3 p^{a_2} q^{1-a_2} + a_4 p^{a_2+a_3} q^{2-(a_2+a_3)} + \dots) \\ &= qF\left(\frac{x}{C_3}\right). \end{aligned} \tag{30}$$

If $x > C_1$,

$$x = \{1, a_2, a_3, \dots\}$$

$$\frac{x - C_1}{C_2} = \{a_2, a_3, \dots\}$$

$$F(x) = pF\left(\frac{x - C_1}{C_2}\right) + q \tag{31}$$

$$\begin{aligned} F(x) &= q(1 + a_2 p + a_3 p^{1+a_2} q^{2-(1+a_2)} + a_4 p^{1+a_2+a_3} q^{3-(1+a_2+a_3)} + \dots) \\ &= pq(a_2 + a_3 p^{a_2} q^{2-(1+a_2)} + \dots) + q \\ &= pF\left(\frac{x - C_1}{C_2}\right) + q. \end{aligned}$$

When $p = C_2$ and $q = C_3$, (27) gives

$$F(x) = \frac{C_3}{C_1} x.$$

Then, if $x < C_1$, (29) becomes

$$F(x) = qF\left(\frac{x}{C_3}\right),$$

$$F(x) = \frac{C_3}{C_1} x = q \frac{x}{C_1} = qF\left(\frac{x}{C_3}\right). \tag{32}$$

For $x > C_1$,

$$F(x) = pF\left(\frac{x - C_1}{C_2}\right) + q, \quad (33)$$

$$F(x) = \frac{C_3 x}{C_1} = \frac{p}{C_2} \frac{C_3 x}{C_1} - \frac{C_3}{C_1} \frac{C_1}{C_2} + C_3 = pF\left(\frac{x - C_1}{C_2}\right) + q.$$

Equations (28) also check with (29) but since those equations are only approximate, the agreement is, of course, only approximate. The interested reader may verify this by using numerical examples.

The distributions discussed above are all based on the motion of a particle confined to an interval. The constraints given in (2) assure that the particle never leaves that interval. One other case merits mention here, viz. that of a particle moving on the positive real axis. Although the results are well known, they are included here in the interest of completeness. To achieve unrestricted motion to the right, the constraints become

$$\begin{aligned} C_1 &= 0, \\ C_2 &> 1, \\ 0 &< C_3 < 1. \end{aligned} \quad (34)$$

Then $x \rightarrow C_2 x$ with probability p and $x \rightarrow C_3 x$ with probability q . The initial point, x_0 , cannot be taken as zero for then the particle would never leave the origin.

For small n , determination of F_n is the same as before. Repeated application of (21) yields an equation involving only F_0 at 2^n arguments and then it remains but to check each argument relative to x_0 .

For n large, another method is available. First rewrite the expression for x_n as follows

$$\begin{aligned} x_n &= x_{n-1} + (C_2 - 1)x_{n-1} \quad \text{with probability } p \\ x_n &= x_{n-1} + (C_3 - 1)x_{n-1} \quad \text{with probability } q. \end{aligned}$$

Then

$$\frac{x_n - x_{n-1}}{x_{n-1}} = \begin{cases} (C_2 - 1) & \text{with probability } p \\ (C_3 - 1) & \text{with probability } q. \end{cases}$$

If ξ is the random variable that takes on the two values, $(C_2 - 1)$ and $(C_3 - 1)$ with probabilities p and q respectively, we have

$$\sum_{i=1}^n \frac{x_i - x_{i-1}}{x_{i-1}} = \sum_{i=1}^n \xi_i. \quad (35)$$

The first and second moments of ξ are

$$m = E(\xi) = pC_2 + qC_3 - 1$$

$$\begin{aligned} E(\xi^2) &= p(C_2^2 - 2C_2 + 1) + q(C_3^2 - 2C_3 + 1) \\ &= pC_2^2 + qC_3^2 - 2(pC_2 + qC_3) + 1 \end{aligned} \quad (36)$$

$$\sigma^2 = E(\xi^2) - (E(\xi))^2 = pq(C_2 - C_3)^2.$$

The right side of (35) is just the sum of n independent, identically distributed random variables. Using standard techniques

from the central limit theorem, the log-normal distribution for x_n (n large) is easily achieved.

$$F_n(x) = \Pr\{x_n \leq x\} \approx \frac{1}{\sqrt{2\pi n} \sigma} \int_0^x \frac{x_0}{t} e^{-(\ln \frac{t}{x_0} - nm)^2 / 2n\sigma^2} dt, \quad (37)$$

where m and σ are given by (36).

4. Moments.

The moments for the distributions of particle positions in the random walk treated here are easily derivable in exact form. The distribution functions do not appear in the derivation nor do the combinations of constraints. The only constraints are those given by (2).

In order to study the moment problem, it is convenient to cast the problem differently. Let

$$x_n = (C_1 + C_2 x_{n-1})\theta_n + C_3 x_{n-1}(1 - \theta_n) \quad (38)$$

where the random variable θ_i is given by

$$\begin{aligned} &= 1 \text{ with probability } p \\ \theta_i &= 0 \text{ with probability } q. \end{aligned} \quad (39)$$

Then

$$x_n = u_n + v_n x_{n-1} \quad (40)$$

where

$$\begin{aligned} u_n &= C_1 \theta_n \\ v_n &= C_2 \theta_n + C_3 (1 - \theta_n). \end{aligned} \quad (41)$$

It is convenient to introduce a table of moments for the u 's and v 's. Equations (39) and (41) give

$$\begin{aligned} E(u_n) &= pC_1 \\ E(v_n) &= pC_2 + qC_3 = K \\ E(u_n v_n) &= pC_1 C_2 \\ E(u_n^2) &= pC_1^2 \\ E(v_n^2) &= pC_2^2 + qC_3^2 = K_2. \end{aligned} \quad (42)$$

Now repeated application of (38) yields an expression for x_n in terms of u 's, v 's, and x_0 .

$$\begin{aligned}
 x_n &= u_n + v_n x_{n-1} \\
 &= u_n + v_n (u_{n-1} + v_{n-1} x_{n-2}) \\
 &\quad \vdots \\
 &= u_n + \sum_{j=1}^{n-1} (u_{n-j} \prod_{k=0}^{j-1} v_{n-k}) + x_0 \prod_{\ell=0}^{n-1} v_{n-\ell}.
 \end{aligned} \tag{43}$$

Applying (42) to (43) gives for the mean of x_n ,

$$\begin{aligned}
 E(x_n) &= E(u_n) + \\
 &\quad \sum_{j=1}^{n-1} E(u_{n-j}) \prod_{k=0}^{j-1} E(v_{n-k}) + x_0 \prod_{\ell=0}^{n-1} E(v_{n-\ell}) \\
 &= E(u_n) + E(u_n) \sum_{j=1}^{n-1} (E(v_j))^j + x_0 \prod_{\ell=0}^{n-1} E(v_{n-\ell}) \\
 &= pC_1 + pC_1 \sum_{j=1}^{n-1} K^j + x_0 K^n \\
 &= pC_1 (1 + K \frac{(1 - K^{n-1})}{(1 - K)}) + x_0 K^n \\
 &= pC_1 \frac{(1 - K^n)}{(1 - K)} + x_0 K^n.
 \end{aligned} \tag{44}$$

To find the second moment, we first square the right side of (43) and then apply (42).

$$\begin{aligned}
 x_n^2 &= u_n^2 + \left[\sum_{j=1}^{n-1} (u_{n-j} \prod_{k=0}^{j-1} v_{n-k}) \right]^2 + x_0^2 \prod_{\ell=0}^{n-1} v_{n-\ell}^2 + \\
 &\quad 2 \left\{ u_n \sum_{j=1}^{n-1} (u_{n-j} \prod_{k=0}^{j-1} v_{n-k}) + x_0 u_n \prod_{\ell=0}^{n-1} v_{n-\ell} + \right. \\
 &\quad \left. x_0 \prod_{\ell=0}^{n-1} v_{n-\ell} \sum_{j=1}^{n-1} (u_{n-j} \prod_{k=0}^{j-1} v_{n-k}) \right\}.
 \end{aligned} \tag{45}$$

The expectations of the individual terms are

$$E(u_n^2) = pC_1^2,$$

$$E\left\{\left[\sum_{j=1}^{n-1} \left(u_{n-j} \prod_{k=0}^{j-1} v_{n-k}\right)\right]^2\right\} =$$

$$E\left\{\sum_{j=1}^{n-1} \sum_{\ell=1}^{n-1} u_{n-j} u_{n-\ell} \prod_{k=0}^{j-1} v_{n-k} \prod_{m=0}^{\ell-1} v_{n-m}\right\} =$$

$$pC_1^2 K_2 \frac{(1 - K_2^{n-1})}{(1 - K_2)} + \frac{2pC_1^2 C_2}{1 - K} \sum_{i=1}^{n-2} K_2^i (1 - K^{n-1-i}),$$

(46)

$$E\left\{x_O^2 \prod_{\ell=0}^{n-1} v_{n-\ell}^2\right\} = x_O^2 K_2^n,$$

$$E\left\{2u_n \sum_{j=1}^{n-1} \left(u_{n-j} \prod_{k=0}^{j-1} v_{n-k}\right)\right\} = 2(pC_1)^2 C_2 \frac{(1 - K^{n-1})}{(1 - K)},$$

$$E\left\{2x_O u_n \prod_{\ell=0}^{n-1} v_{n-\ell}\right\} = 2x_O pC_1 C_2 K^{n-1},$$

$$E\left\{2x_O \prod_{\ell=0}^{n-1} v_{n-\ell} \sum_{j=1}^{n-1} \left(u_{n-j} \prod_{k=0}^{j-1} v_{n-k}\right)\right\} = 2x_O pC_1 C_2 K_2 \frac{(K^{n-1} - K_2^{n-1})}{(K - K_2)}.$$

Collecting terms,

$$E(x_n^2) = pC_1^2 \frac{(1 - K_2^n)}{(1 - K_2)} + 2 \frac{(pC_1)^2 C_2}{(1 - K)} \left[\frac{(1 - K_2^{n-1})}{(1 - K)} - \frac{K(K^{n-1} - K_2^{n-1})}{(K - K_2)} \right] +$$

(47)

$$x_O^2 K_2^n + 2x_O pC_1 C_2 \frac{(K^n - K_2^n)}{(K - K_2)}.$$

Higher moments are derivable in exactly the same way as $E(x_n)$ and $E(x_n^2)$ but no purpose is served in dwelling further on those moments.

The moments for the limiting distributions may be obtained immediately from (44) and (47). If x is the random variable denoting position after an infinite number of steps,

$$\begin{aligned} E(x) &= \frac{pC_1}{1-K} \\ E(x^2) &= \frac{pC_1^2}{1-K_2} \left(1 + \frac{2pC_2}{1-K}\right) \\ \sigma^2 &= E(x^2) - (E(x))^2 = \frac{pqC_1^2(1-C_3)^2}{(1-K)^2(1-K_2)} \end{aligned} \quad (48)$$

The moments derived here provide a check for at least one of the distributions derived in Section 3. When $p = C_2$ and $q = C_3$, equation (27) gives

$$\begin{aligned} F(x) &= \frac{C_3}{C_1}x \\ E(x) &= \frac{C_3}{C_1} \int_0^{\frac{C_1}{C_3}} x dx \\ &= \frac{C_1}{2C_3} \end{aligned} \quad (49)$$

Equation (48) gives for $E(x)$,

$$\begin{aligned} E(x) &= \frac{pC_1}{1 - C_2^2 - C_3^2} \\ &= \frac{C_2 C_1}{2C_2 C_3} = \frac{C_1}{2C_3} . \end{aligned} \quad (50)$$

Similarly,

$$E(x^2) = \frac{C_3}{C_1} \int_0^{\frac{C_1}{C_3}} x^2 dx = \frac{C_1^2}{3C_3^2} \quad (51)$$

From (48)

$$\begin{aligned} E(x^2) &= \frac{pC_1^2}{1 - C_2^2 - C_3^2} \left(1 + \frac{2C_2^2}{2C_2 C_3} \right) \\ &= \frac{pC_1^2}{3C_2 C_3^2} (C_2 + C_3) \\ &= \frac{C_1^2}{3C_3^2} . \end{aligned} \quad (52)$$

This work on the derivation of moments for distributions of particle positions in the random walk treated here was done simultaneously with some work of J. B. Tysver on the same subject. The results were obtained independently.

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